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INSTANTONS AND RECURSION RELATIONS IN N=2 SUSY GAUGE THEORY‡

MARCO MATONE

Department of Physics "G. Galilei" - Istituto Nazionale di Fisica Nucleare
University of Padova
Via Marzolo, 8 - 35131 Padova, Italy

ABSTRACT

We find the transformation properties of the prepotential $\mathcal{F}$ of $N=2$ SUSY gauge theory with gauge group $SU(2)$. In particular we show that $\mathcal{G}(a)=\pi i\left(\mathcal{F}(a)-\frac{1}{2}a\partial_a\mathcal{F}(a)\right)$ is modular invariant. This function satisfies the non-linear differential equation $(1-\mathcal{G}^2)\,\mathcal{G}''+\frac{1}{4}a\mathcal{G}'^3=0$, implying that the instanton contribution are determined by recursion relations. Finally, we find $u=u(a)$ and give the explicit expression of $\mathcal{F}$ as function of u. These results can be extended to more general cases.

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1. Recently the low-energy limit of $N = 2$ super Yang-Mills theory with gauge group $G = SU(2)$ has been solved exactly [1]. This result has been generalized to $G = SU(n)$ in [2] whereas the large n analysis has been investigated in [3]. Other interesting results concern the generalization to $SO(2n+1)$ [4] and non-locality at the cusp points in moduli spaces [5].

The low-energy effective action S_{eff} is derived from a single holomorphic function $\mathcal{F}(\Phi_k)$ [6]

$$S_{eff} = \frac{1}{4\pi}\mathrm{Im}\left(\int d^2\theta d^2\bar{\theta}\Phi_D^i\overline{\Phi}_i + \frac{1}{2}\int d^2\theta\tau^{ij}W_iW_j\right), \tag{1}$$

where $\Phi_D^i \equiv \partial\mathcal{F}/\partial\Phi_i$ and $\tau^{ij} \equiv \partial^2\mathcal{F}/\partial\Phi_i\partial\Phi_j$. Let us denote by $a_i \equiv \langle\phi^i\rangle$ and $a_D^i \equiv \langle\phi_D^i\rangle$ the vevs of the scalar component of the chiral superfield. For $SU(2)$ the moduli space of quantum vacua, parametrized by $u = \langle\mathrm{tr}\,\phi^2\rangle$, is the Riemann sphere with punctures at $u_1 = -\Lambda, u_2 = \Lambda$ (we will set $\Lambda = 1$) and $u_3 = \infty$ and a $\mathbf{Z}_2$ symmetry acting by $u \leftrightarrow -u$. The asymptotic expansion of the prepotential has the structure [1]

$$\mathcal{F} = \frac{i}{2\pi}a^2\log a^2 + \sum_{k=0}^{\infty}\mathcal{F}_k a^{2-4k}. \tag{2}$$

In [1] the vector (a_D, a) has been considered as a holomorphic section of a flat bundle. In particular in [1] the monodromy properties of $(a_D(u), a(u))$ have been identified with $\Gamma(2)$

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \Longrightarrow \begin{pmatrix} \tilde{a}_D \\ \tilde{a} \end{pmatrix} = M_{u_i}\begin{pmatrix} a_D \\ a \end{pmatrix}, \qquad i = 1, 2, 3, \tag{3}$$

where

$$M_{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}.$$

The asymptotic behaviour of this section, derived in [1], and the geometrical data above completely determine $(a_D(u), a(u))$. In particular the explict expression of the section (a_D, a) has been obtained by first constructing tori parametrized by u and then identifying a suitable meromorphic differential [1].

Before considering the framework of uniformization theory, we find the explicit expression of $\mathcal{F}$ in terms of u. Next we will find the modular properties of $\mathcal{F}$ by solving a linear differential equation which arises from defining properties. We will use uniformization theory in order to explicitly find $u = u(a)$ and to derive the (non-linear) differential equation satisfied by $\mathcal{F}$ as a function of a. This equation furnishes, as expected, recursion relations which determine the instanton contributions to $\mathcal{F}$. Our general formula is in agreement with the results in [7] where the first six terms of the instanton contribution have been computed.

Let us start with the explicit expression of $\mathcal{F}$ as function of u. Let us recall that [1]

$$a_D = \frac{\sqrt{2}}{\pi}\int_1^u \frac{dx\sqrt{x-u}}{\sqrt{x^2-1}}, \qquad a = \frac{\sqrt{2}}{\pi}\int_{-1}^1 \frac{dx\sqrt{x-u}}{\sqrt{x^2-1}}. \tag{4}$$

In order to solve the problem we use the integrability of the 1-differential

$$\eta(u) = a\partial_u a_D - a_D\partial_u a = \frac{1}{\pi^2}\int_1^u dx \int_{-1}^1 dy \frac{y-x}{\sqrt{(x^2-1)(x-u)(y^2-1)(y-u)}}. \tag{5}$$

We have

$$g(u) = \int_1^u dz\eta(z) = \frac{1}{\pi^2}\int_1^u dx\int_{-1}^1 dy \frac{y-x}{\sqrt{(x^2-1)(y^2-1)}} \log\left[\frac{2u-x-y+2\sqrt{(u-x)(u-y)}}{x-y}\right]. \tag{6}$$

On the other hand notice that

$$\partial_u\mathcal{F} = a_D\partial_u a = \frac{1}{2}[\partial_u(aa_D) - \eta(u)],$$

so that, up to an additive constant, we have

$$\mathcal{F}(a(u)) = \frac{1}{2\pi^2}\int_1^u dx\int_{-1}^1 dy \frac{4\sqrt{(x-u)(y-u)} - (y-x)\log\left[\frac{2u-x-y+2\sqrt{(u-x)(u-y)}}{x-y}\right]}{\sqrt{(x^2-1)(y^2-1)}}. \tag{7}$$

Later, in the framework of uniformization theory, we will show that η is a constant (in the u-patch), so that g is proportional to u.

We now find the transformation properties of $\mathcal{F}(a)$. By (15), we have

$$\frac{\partial^2\widetilde{\mathcal{F}}(\tilde{a})}{\partial\tilde{a}^2} = \frac{A\frac{\partial^2\mathcal{F}(a)}{\partial a^2}+B}{C\frac{\partial^2\mathcal{F}(a)}{\partial a^2}+D}, \tag{8}$$

where $\begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \Gamma(2)$ and $\tilde{a} = Ca_D + Da$. On the other hand

$$\frac{\partial^2\widetilde{\mathcal{F}}(\tilde{a})}{\partial\tilde{a}^2} = \left[-\left(\frac{\partial\tilde{a}}{\partial a}\right)^{-3}\frac{\partial^2\tilde{a}}{\partial a^2}\frac{\partial}{\partial a} + \left(\frac{\partial\tilde{a}}{\partial a}\right)^{-2}\frac{\partial^2}{\partial a^2}\right]\widetilde{\mathcal{F}}(\tilde{a}). \tag{9}$$

Eqs.(8) (9) imply that

$$(C\mathcal{F}^{(2)}+D)\partial_a^2\widetilde{\mathcal{F}}(\tilde{a}) - C\mathcal{F}^{(3)}\partial_a\widetilde{\mathcal{F}}(\tilde{a}) - (A\mathcal{F}^{(2)}+B)(C\mathcal{F}^{(2)}+D)^2 = 0, \tag{10}$$

where $\mathcal{F}^{(k)} \equiv \partial_a^k \mathcal{F}(a)$, whose solution is

$$\widetilde{\mathcal{F}}(\tilde{a}) = \mathcal{F}(a) + \frac{AC}{2} a_D^2 + \frac{BD}{2} a^2 + BCaa_D. \tag{11}$$

This means that the function

$$\mathcal{G}(a) = \pi i \left(\mathcal{F}(a) - \frac{1}{2} a \partial_a \mathcal{F}(a) \right) = -\frac{\pi i}{2} g(u), \tag{12}$$

is modular invariant, that is

$$\widetilde{\mathcal{G}}(\tilde{a}) = \mathcal{G}(a). \tag{13}$$

By (2) we have asymptotically

$$\mathcal{G} = \sum_{k=0}^{\infty} \mathcal{G}_k a^{2-4k}, \qquad \mathcal{G}_0 = \frac{1}{2}, \quad \mathcal{G}_k = 2\pi i k \mathcal{F}_k. \tag{14}$$

2. In order to find $u = u(a)$ and $\mathcal{F}$ as function of a, we need few facts about uniformization theory. Let us denote by $\widehat{\mathbf{C}} \equiv \mathbf{C} \cup \{\infty\}$ the Riemann sphere and by H the upper half plane endowed with the Poincaré metric $ds^2 = |dz|^2/(\operatorname{Im} z)^2$. It is well known that n-punctured spheres $\Sigma_n \equiv \widehat{\mathbf{C}} \backslash \{u_1, \ldots, u_n\}$, $n \geq 3$, can be represented as H/Γ with $\Gamma \subset PSL(2, \mathbf{R})$ a parabolic (i.e. with $|\operatorname{tr} \gamma| = 2$, $\gamma \in \Gamma$) Fuchsian group. The map $J_H : H \longrightarrow \Sigma_n$ has the property $J_H(\gamma \cdot z) = J_H(z)$, where $\gamma \cdot z = (Az + B)/(Cz + D)$, $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$. It follows that after winding around nontrivial loops the inverse map transforms as

$$J_H^{-1}(u) \longrightarrow \widetilde{J}_H^{-1}(u) = \frac{AJ_H^{-1}(u) + B}{CJ_H^{-1}(u) + D}. \tag{15}$$

The projection of the Poincaré metric onto $\Sigma_n \cong H/\Gamma$ is

$$ds^2 = e^{\varphi} |du|^2 = \frac{|J_H^{-1}(u)'|^2}{(\operatorname{Im} J_H^{-1}(u))^2} |du|^2, \tag{16}$$

which is invariant under $SL(2, \mathbf{R})$ fractional transformations of J_H^{-1}. The fact that e^{φ} has constant curvature -1 means that φ satisfies the Liouville equation

$$\partial_u \partial_{\bar{u}} \varphi = \frac{e^{\varphi}}{2}. \tag{17}$$

Near a puncture we have $\varphi \sim -\log \left(|u - u_i|^2 \log^2 |u - u_i| \right)$. For the Liouville stress tensor we have the following equivalent expressions

$$T(u) = \partial_u \partial_u \varphi - \frac{1}{2} (\partial_u \varphi)^2 = \left\{ J_H^{-1}, u \right\} = \sum_{i=1}^{n-1} \left(\frac{1}{2(u - u_i)^2} + \frac{c_i}{u - u_i} \right). \tag{18}$$

where $\left\{J_H^{-1}, u\right\}$ denotes the Schwarzian derivative of J_H^{-1} and the c_i's, called accessory parameters, satisfy the constraints

$$\sum_{i=1}^{n-1} c_i = 0, \qquad \sum_{i=1}^{n-1} c_i u_i = 1 - \frac{n}{2}. \tag{19}$$

Let us now consider the covariant operators introduced in the formulation of the KdV equation in higher genus [8]. We use $1/J_H^{-1\prime}$ as covariantizing polymorphic vector field [9]

$$\mathcal{S}^{(2k+1)}_{J_H^{-1}} = (2k+1) {J_H^{-1\prime}}^k \partial_u \frac{1}{J_H^{-1\prime}} \partial_u \frac{1}{J_H^{-1\prime}} \ldots \partial_u \frac{1}{J_H^{-1\prime}} \partial_u {J_H^{-1\prime}}^k, \tag{20}$$

where the number of derivatives is $2k+1$ and $' \equiv \partial_u$. Univalence of J_H^{-1} implies holomorphicity of $\mathcal{S}^{(2k+1)}_{J_H^{-1}}$. An interesting property of the equation $\mathcal{S}^{(2k+1)}_{J_H^{-1}} \cdot \psi = 0$ is that its projection on H reduces to the trivial equation $(2k+1){z'}^{k+1}\partial_z^{2k+1}\widetilde{\psi} = 0$, where $z = J_H^{-1}(u)$. Operators $\mathcal{S}^{(2k+1)}_{J_H^{-1}}$ are covariant, holomorphic and $SL(2, \mathbf{C})$ invariant, which by (15) implies singlevaluedness of $\mathcal{S}^{(2k+1)}_{J_H^{-1}}$. Furthermore, Möbius invariance of the Schwarzian derivative implies that $\mathcal{S}^{(2k+1)}_{J_H^{-1}}$ depends on J_H^{-1} only through the stress tensor (18) and its derivatives. For $k = 1/2$, we have the *uniformizing equation*

$$\left(J_H^{-1\prime}\right)^{\frac{1}{2}} \partial_u \frac{1}{J_H^{-1\prime}} \partial_u \left(J_H^{-1\prime}\right)^{\frac{1}{2}} \cdot \psi = \left(\partial^2 + \frac{T}{2}\right) \cdot \psi = 0, \tag{21}$$

that, by construction, has the two linearly independent solutions

$$\psi_1 = \left(J_H^{-1\prime}\right)^{-\frac{1}{2}} J_H^{-1}, \qquad \psi_2 = \left(J_H^{-1\prime}\right)^{-\frac{1}{2}}, \tag{22}$$

so that

$$J_H^{-1} = \psi_1/\psi_2. \tag{23}$$

By (15) and (22) it follows that

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \widetilde{\psi}_1 \\ \widetilde{\psi}_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{24}$$

In the case of $\Sigma_3 \cong H/\Gamma(2)$, Eq.(19) gives $c_1 = -c_2 = 1/4$ and the uniformizing equation (21) becomes[1]

$$\left(\partial_u^2 + \frac{3+u^2}{4(1-u^2)^2}\right)\psi = 0, \tag{25}$$

[1] This equation has been considered also in [10].

which is solved by Legendre functions

$$\psi_1 = \sqrt{1-u^2} P_{-1/2}, \qquad \psi_2 = \sqrt{1-u^2} Q_{-1/2}. \tag{26}$$

These solutions define a holomorphic section that by (24) has monodromy $\Gamma(2)$.

In order to find (a, a_D) we observe that by (22) ψ_1 and ψ_2 are (polymorphic) $-1/2$-differentials whereas both a_D and a are 0-differentials. This fact and the asymptotic behaviour of (a_D, a) given in [1] imply that

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \sqrt{1-u^2}\partial_u a_D \\ \sqrt{1-u^2}\partial_u a \end{pmatrix}, \tag{27}$$

where $\sqrt{1-u^2}$ is considered as a $-3/2$-differential. Comparing with (26) we get (4).

3. By Eqs.(25) and (27) it follows that a_D and a are solutions of the third-order equation

$$\left(\partial_u^2 + \frac{3+u^2}{4(1-u^2)^2}\right)\sqrt{1-u^2}\partial_u \phi = 0. \tag{28}$$

Let us consider some aspects of this equation. First of all note that, as observed in [7],

$$\left(\partial_u^2 + \frac{3+u^2}{4(1-u^2)^2}\right)\sqrt{1-u^2}\partial_u \phi = \frac{1}{\sqrt{1-u^2}}\partial_u \left[(1-u^2)\partial_u^2 - \frac{1}{4}\right]\phi = 0. \tag{29}$$

It follows that $\left[(1-u^2)\partial_u^2 - \frac{1}{4}\right]\phi = c$ with c a constant. A check shows that a_D and a in (4) satisfy this equation with $c = 0$

$$\left[(1-u^2)\partial_u^2 - \frac{1}{4}\right] a_D = \left[(1-u^2)\partial_u^2 - \frac{1}{4}\right] a = 0. \tag{30}$$

As noticed in [7], this explains also why, despite of the fact that a and a_D satisfy the third-order differential equation (28), they have two-dimensional monodromy. Eq.(30) is the crucial one to find $u = u(a)$ and to determine the instanton contributions. In our framework the problem of finding the form of $\mathcal{F}$ as a function of a is equivalent to the following general basic problem which is of interest also from a mathematical point of view:

Given a second-order differential equation with solutions ψ_1 and ψ_2 find the function $\mathcal{F}_1(\psi_1)$ ($\mathcal{F}_2(\psi_2)$) such that $\psi_2 = \partial\mathcal{F}_1/\partial\psi_2$ ($\psi_1 = \partial\mathcal{F}_2/\partial\psi_2$).

We show that such a function satisfies a non-linear differential equation. The first step is to observe that by (30) it follows that

$$a a_D' - a_D a' = c. \tag{31}$$

Since (a_D, a) are (polymorphic) 0-differentials, it follows that in changing patch the constant c in (31) is multiplied by the Jacobian of the coordinate transformation. Another equivalent way to see this, is to notice that Eq.(30) gets a first derivative under a coordinate transformation. Therefore in another patch the r.h.s. of (31) is no longer a constant. As we have seen, covariance of the equation such has

$$(\partial_z^2 + F(z)/2)\psi(z) = 0,$$

is ensured if and only if ψ transforms as a $-1/2$-differential and F as a Schwarzian derivative. In terms of the solutions ψ_1, ψ_2 one can construct the 0-differential $\psi_1'\psi_2 - \psi_1\psi_2'$ that, by the structure of the equation, is just a constant c. In another patch we have $(\partial_w^2 + \widetilde{F}(w)/2)\widetilde{\psi}(w) = 0$, so that $\psi_1(z)\partial_z\psi_2(z) - \psi_2(z)\partial_z\psi_1(z) = \widetilde{\psi}_1(w)\partial_w\widetilde{\psi}_2(w) - \widetilde{\psi}_2(w)\partial_w\widetilde{\psi}_1(w) = c$.

This discussion shows that flatness of a_D and a is the reason of the reduction mechanism from the third-order to second-order equation.

By (5) (6) (12) and (31) it follows that

$$Au + B = \mathcal{G}(a), \tag{32}$$

where B is a constant which we will show to be zero. To determine the constant A, we note that asymptotically $a \sim \sqrt{2u}$, therefore by (14) one has $A = 1$. By (4) and (32) it follows that

$$a_D = \frac{\sqrt{2}}{\pi}\int_1^{\mathcal{G}(a)+B} \frac{dx\sqrt{x - \mathcal{G}(a) - B}}{\sqrt{x^2-1}}, \qquad a = \frac{\sqrt{2}}{\pi}\int_{-1}^{1} \frac{dx\sqrt{x - \mathcal{G}(a) - B}}{\sqrt{x^2-1}}. \tag{33}$$

Apparently to solve these two equivalent integro-differential equations seems a difficult task. However we can use the following trick. First notice that

$$\left[(1-u^2)\partial_u^2 - \frac{1}{4}\right]\phi = 0 = \left\{\left[1 - (\mathcal{G}+B)^2\right]\left(\mathcal{G}'\partial_a^2 - \mathcal{G}''\partial_a\right) - \frac{1}{4}\mathcal{G}'^3\right\}\phi = 0, \tag{34}$$

where now $' \equiv \partial_a$. Then, since $\phi = a$ (or equivalently $\phi = a_D = \partial_a\mathcal{F}$) is a solution of (34), it follows that $\mathcal{G}(a)$ satisfies the non-linear differential equation $[1 - (\mathcal{G}+B)^2]\,\mathcal{G}'' + \frac{1}{4}a\mathcal{G}'^3 = 0$. Inserting the expansion (14) one can check that the only way to compensate the $a^{-2(2k+1)}$ terms is to set $B = 0$. Therefore

$$\left(1 - \mathcal{G}^2\right)\mathcal{G}'' + \frac{1}{4}a\mathcal{G}'^3 = 0, \tag{35}$$

which is equivalent to the following recursion relations for the instanton contribution (recall that $\mathcal{G} = 2\pi i k\mathcal{F}_k$)

$$\mathcal{G}_{n+1} = \frac{1}{8\mathcal{G}_0^2(n+1)^2}.$$

$$\cdot \left\{ (2n-1)(4n-1)\mathcal{G}_n + 2\mathcal{G}_0 \sum_{k=0}^{n-1} \mathcal{G}_{n-k}\mathcal{G}_{k+1} c(k,n) - 2\sum_{j=0}^{n-1}\sum_{k=0}^{j+1} \mathcal{G}_{n-j}\mathcal{G}_{j+1-k}\mathcal{G}_k d(j,k,n) \right\}, \quad (36)$$

where $n \geq 0$, $\mathcal{G}_0 = 1/2$ and

$$c(k,n) = 2k(n-k-1) + n - 1, \qquad d(j,k,n) = [2(n-j)-1][2n-3j-1+2k(j-k+1)].$$

The first few terms are $\mathcal{G}_0 = \frac{1}{2}$, $\mathcal{G}_1 = \frac{1}{2^2}$, $\mathcal{G}_2 = \frac{5}{2^6}$, $\mathcal{G}_3 = \frac{9}{2^7}$, in agreement[2] with the results in [7] where the first terms of the instanton contribution have been computed by first inverting $a(u)$ as a series for large a/Λ and then inserting this in a_D.

Finally let us notice that the inverse of $a = a(u)$ is

$$u = \mathcal{G}(a), \quad (37)$$

and

$$a a_D' - a_D a' = \frac{2i}{\pi}, \quad (38)$$

which is useful to explicitly determine the critical curve on which $\mathrm{Im}\, a_D/a = 0$, whose structure has been considered in [1][11][12].

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[2] Notice that we are using different normalizations, thus to compare with $\mathcal{F}_k^{KLT}$ in [7] one should check the invariance of the quantity $\frac{\mathcal{F}_k}{\mathcal{F}_k^{KLT}}\frac{\mathcal{F}_{k+1}^{KLT}}{\mathcal{F}_{k+1}}$.

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